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Group-theoretical analysis of aperiodic tilings from projections of higher-dimensional lattices B_n

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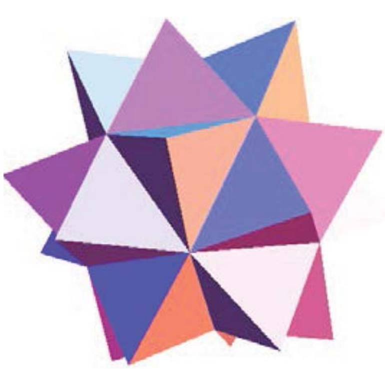
A group-theoretical discussion on the hypercubic lattice described by the affine Coxeter–Weyl group $W_a(B_n)$ is presented. When the lattice is projected onto the Coxeter plane it is noted that the maximal dihedral subgroup D_h of $W(B_n)$ with $h = 2n$ representing the Coxeter number describes the h-fold symmetric aperiodic tilings. Higher-dimensional cubic lattices are explicitly constructed for $n = 4, 5, 6$. Their rank-3 Coxeter subgroups and maximal dihedral subgroups are identified. It is explicitly shown that when their Voronoi cells are decomposed under the respective rank-3 subgroups $W(A_3)$, $W(H_2) \times W(A_1)$ and $W(H_3)$ one obtains the rhombic dodecahedron, rhombic icosahedron and rhombic triacontahedron, respectively. Projection of the lattice B_4 onto the Coxeter plane represents a model for quasicrystal structure with eightfold symmetry. The B_5 lattice is used to describe both fivefold and tenfold symmetries. The lattice B_6 can describe aperiodic tilings with 12-fold symmetry as well as a three-dimensional icosahedral symmetry depending on the choice of subspace of projections. The novel structures from the projected sets of lattice points are compatible with the available experimental data.

1. Introduction

Quasicrystallography gained enormous impetus after the first discovery of the icosahedral quasicrystal by D. Shechtman (Shechtman *et al.*, 1984). Its point symmetry can be described by the Coxeter group $W(H_3)$ representing the icosahedral symmetry of order 120. Recent developments indicate that the quasicrystals exhibit fivefold, eightfold, tenfold, 12-fold and 18-fold symmetries. For a general exposition we refer the reader to the literature on quasicrystallography (Janot, 2012; Di Vincenzo & Steinhardt, 1991; Senechal, 1995; Steurer, 2004; Tsai, 2008). In a recent paper (Lubin *et al.*, 2012) it was reported that a quasicrystallographic structure with 36-fold symmetry is possible. These predictions imply that no limitation exists on the order of the planar point symmetry of the quasicrystallography described by the dihedral groups. That reminds us of the classification of the Coxeter–Weyl groups with different Coxeter numbers h (Coxeter, 1951; Coxeter & Moser, 1965; Bourbaki, 1968; Humphreys, 1990). Every Coxeter–Weyl group has a dihedral subgroup D_h of order $2h$. This paper attempts to illustrate the relations between the group-theoretical structures of the affine Coxeter groups $W_a(B_n)$ and the h-fold symmetric quasicrystallography obtained from higher-dimensional cubic lattices by orthogonal projections. A general projection technique of the higher-dimensional cubic lattice is prescribed by Duneau & Katz (1985) but with no detailed group-theoretical discussion on the symmetries of the lattices. For an earlier work see also de Wolff (1974).



There have been several other approaches by using affine extensions of the noncrystallographic Coxeter groups (Moody & Patera, 1993; Patera & Twarock, 2002; Dechant *et al.*, 2012, 2013) for the description of fivefold symmetric quasicrystals as well as quasicrystals with icosahedral symmetry. There is yet another approach for the description of quasicrystal structures. This is a set-theoretic approach initiated by Yves Meyer (Meyer, 1972, 1995) and later developed by Robert V. Moody (Moody, 1995, 2000) in the name of *Model Set*.

The Lie groups derived from the root systems of the Coxeter–Weyl groups are well known by the high-energy physicists. Predictions of the standard model of high-energy physics described by the Lie group $SU(3) \times SU(2) \times U(1)$ (Weinberg, 1967; Salam, 1968; Fritzsch *et al.*, 1973) are heavily based on the Coxeter–Weyl group $W(A_2) \times W(A_1)$. The skeletons of the Grand Unified theories, $SU(5) \approx E_4$ (Georgi & Glashow, 1974), $SO(10) \approx E_5$ (Fritzsch & Minkowski, 1975) and the exceptional group E_6 (Gürsey *et al.*, 1976) are the respective Coxeter–Weyl groups $W(A_4)$, $W(D_5)$ and $W(E_6)$. It is expected that some of the Coxeter–Weyl groups with a Coxeter number h may also play an important role in the study of quasicrystallography. The argument is based on the following line of thought.

Any Coxeter group with a Coxeter number h has a maximal dihedral subgroup D_h of order $2h$ which acts in the Coxeter plane. [It is unfortunate that the same notation is also used for the Coxeter–Weyl group $W(D_n)$]. In two recent papers (Koca, Koca & Koc, 2014*a*; Koca, Koca & Al-Sawafi, 2014) we have proposed that a quasicrystallographic structure with h-fold symmetry can be determined by projections of the higher-dimensional lattices onto the relevant Coxeter planes. The Coxeter groups are naturally characterized by some integers known as Coxeter exponents (Coxeter, 1951; Humphreys, 1990). In this paper we study the general structure of the root lattice of the affine Coxeter group $W_a(B_n)$. It is the simple cubic lattice in nD Euclidean space with the point symmetry determined by the Coxeter–Weyl group $W(B_n)$ of order $2^n n!$. The projection of the five-dimensional cubic lattice onto a plane and the projection of the six-dimensional cubic lattice into a three-dimensional subspace have been studied previously without using Coxeter group techniques (Duneau & Katz, 1985; de Bruijn, 1981). Projections of some four-dimensional root lattices have also been studied previously (Kramer & Neri, 1984; Baake, Kramer *et al.*, 1990; Baake, Joseph *et al.*, 1990).

The group $W(B_n)$ itself can be regarded as an extension of the elementary abelian group 2^n by its permutation group S_n. The paper is organized as follows. In §2 we study the general structure of the Coxeter–Weyl group $W(B_n)$ with some emphasis on its maximal subgroups which could be useful for the projections of the hypercubic lattices. §3 deals with the study of the rank-3 subgroups of the Coxeter–Weyl groups $W(B_4)$, $W(B_5)$ and $W(B_6)$ and projections of some of their polytopes into three-dimensional Euclidean spaces with different residual symmetries. The projection of the Voronoi cell of a higher-dimensional lattice plays a crucial role in the description of the quasicrystallographic structures in three dimensions and two dimensions. Its structure for the $W(B_n)$ lattices will be pointed out. §4 is devoted to the projection techniques of the lattices onto the Coxeter planes and the projection of the six-dimensional cubic lattice into three-dimensional subspace with icosahedral symmetry. Our predictions are compared with experimental data in §5 and some concluding remarks are added.

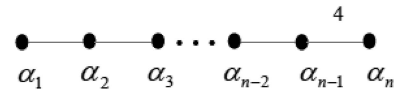


Figure 1
Coxeter–Dynkin diagram of the Coxeter–Weyl group $W(B_n)$.

2. The Coxeter–Weyl group $W(B_n)$ and its maximal subgroups

The classification of the Coxeter–Weyl groups is well known (Coxeter, 1951; Coxeter & Moser, 1965; Bourbaki, 1968; Humphreys, 1990). It includes an infinite series of crystallographic groups A_n, B_n, C_n, D_n, and a finite number of crystallographic exceptional groups G_2, F_4, E_6, E_7, E_8. In addition to the above crystallographic groups there are an infinite number of noncrystallographic dihedral Coxeter groups $I_2(h)$ $(h \geq 5, h \neq 6)$ and the two rank-3 and rank-4 noncrystallographic Coxeter groups $W(H_3)$ and $W(H_4)$, respectively. In this section we will be interested in the group-theoretical structures of the Coxeter–Weyl group $W(B_n)$ of rank n. It is represented by the Coxeter–Dynkin diagram shown in Fig. 1.

From left to right the nodes denote the long simple roots α_i $(i = 1, 2, \ldots, n-1)$ with norm $\sqrt{2}$ and the last simple root α_n is the short root with norm 1. They represent linearly independent vectors in n-dimensional Euclidean space. The angle between any two adjacent simple roots with norm $\sqrt{2}$ is 120° and the angle between the last two simple roots is 135°. Any two disconnected roots are orthogonal to each other. The nodes also denote the reflection generators r_i whose action on an arbitrary vector Λ in the n-dimensional Euclidean space is given by

$$r_i \Lambda = \Lambda - \frac{2(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i. \tag{1}$$

It is useful to work in the dual space of the root space represented by the weight vectors ω_i defined by

$$\left(\omega_i, \frac{2\alpha_j}{(\alpha_j, \alpha_j)}\right) = \delta_{ij}.$$

When the direct space is associated with the root space the reciprocal lattice is associated with the weight space. The Cartan matrix of the root space (Gram matrix) and the metric tensor in the dual space are defined, respectively, by the matrix elements

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \; G_{ij} = (\omega_i, \omega_j) = (A^{-1})_{ij} \frac{(\alpha_j, \alpha_j)}{2}. \tag{2}$$

Let l_i $(i = 1, 2, \ldots, n)$ be the set of orthonormal vectors $(l_i, l_j) = \delta_{ij}$ in n-dimensional Euclidean space. The simple roots of $W(B_n)$ can be written as

$$\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, \ldots, \alpha_{n-1} = l_{n-1} - l_n, \alpha_n = l_n. \quad (3)$$

The root system consists of two sets, one set with $2n$ short roots $\pm l_i$ and the other with $2n(n-1)$ long roots $\pm l_i \pm l_j\, (i \neq j)$. Reflection generators r_i act on the unit vectors as follows:

$$r_1 : l_1 \leftrightarrow l_2, r_2 : l_2 \leftrightarrow l_3, \ldots, r_{n-1} : l_{n-1} \leftrightarrow l_n, r_n : l_n \rightarrow -l_n. \quad (4)$$

The generators r_i generate the Coxeter–Weyl group $W(B_n)$. Equation (4) implies that the reflection generators leave the other unit vectors (not shown) invariant. The weight vectors can be determined from $\omega_i = (A^{-1})_{ij}\alpha_j$ as

$$\omega_1 = l_1, \omega_2 = l_1 + l_2, \ldots, \omega_{n-1} = l_1 + l_2 + \ldots + l_{n-1},$$
$$\omega_n = \frac{1}{2}(l_1 + l_2 + \ldots + l_n). \quad (5)$$

Any highest weight vector (Slansky, 1981) in the weight space can be written as $\Lambda = a_1\omega_1 + a_2\omega_2 + \ldots + a_n\omega_n \equiv (a_1, a_2, \ldots, a_n)$ with integer coefficients $a_i \geq 0$. We will delete the commas between the integers as long as an integer does not exceed 9. An orbit of Λ under the Coxeter–Weyl group $W(B_n)$ will be denoted by $W(B_n)\Lambda \equiv (a_1 a_2 \ldots a_n)_{B_n}$. With this notation, *e.g.*, the orbits $(100\ldots0)_{B_n} = \pm l_i$ and $(010\ldots0)_{B_n} = \pm l_i \pm l_j$ $(i \neq j)$ represent the sets of short roots and long roots, respectively. The orbit $(00\ldots01)_{B_n} = \frac{1}{2}(\pm l_1 \pm l_2 \pm \ldots \pm l_n)$ represents the vertices of a cube in n-dimensional Euclidean space. Before we discuss the maximal subgroups of the group $W(B_n)$ we point out that the same Coxeter–Weyl group defines another Coxeter–Dynkin diagram C_n where the short and long roots are represented by the roots $\alpha_1 = l_1 - l_2$, $\alpha_2 = l_2 - l_3, \ldots, \alpha_{n-1} = l_{n-1} - l_n$, $\alpha_n = 2l_n$. Here the short and long roots of $W(B_n)$ are interchanged but the symmetry group is the same as defined in equation (4). Certain maximal subgroups of $W(B_n)$ can be useful in the study of quasicrystals. One of the maximal subgroups of $W(B_n)$ is the Coxeter–Weyl group $W(D_n)$ with the Coxeter–Dynkin diagram given in Fig. 2. It consists of only long roots of norm $\sqrt{2}$.

The group $W(D_n)$ is a maximal subgroup with an order of $2^{n-1}n!$ in the group $W(B_n)$ with an index 2. Denote by r_i' the reflection generators of $W(D_n)$ and the simple roots by

$$\alpha_1' = l_1 - l_2, \alpha_2' = l_2 - l_3, \ldots, \alpha_{n-1}' = l_{n-1} - l_n, \alpha_n' = l_{n-1} + l_n. \quad (6)$$

The first $n - 1$ simple roots are identical to the simple roots of $W(B_n)$ except the last one. The generators of $W(D_n)$ transform

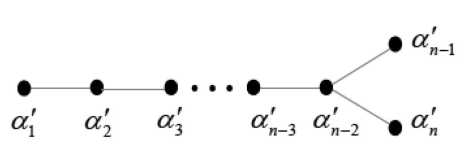


Figure 2
Coxeter–Dynkin diagram of $W(D_n)$.

the unit vectors as in equation (4) but the last generator differs in its action:

$$r_1' : l_1 \leftrightarrow l_2, r_2' : l_2 \leftrightarrow l_3, \ldots, r_{n-1}' : l_{n-1} \leftrightarrow l_n, r_n' : l_{n-1} \leftrightarrow -l_n. \quad (7)$$

This implies that one can identify $r_i{}' = r_i$ (for $i = 1, 2, \ldots, n-1$) and we note that $r_n' = r_n r_{n-1} r_n$. The Coxeter–Dynkin diagram of $W(D_n)$ has a diagram symmetry $\gamma : \alpha_{n-1}' \leftrightarrow \alpha_n'$ which transforms $\gamma : l_n \rightarrow -l_n$ leaving the other unit vectors invariant and it can be identified with r_n. Therefore the automorphism group of D_n is $\mathrm{Aut}(D_n) \approx W(D_n) : \mathbb{Z}_2 \approx W(B_n)$ where the group $\mathbb{Z}_2$ is generated by γ. We note that there is one exception to this general result: the automorphism group of D_4 is larger than $W(B_4)$ since its Dynkin diagram symmetry is isomorphic to the symmetry group $S_3 \approx D_3$ of order 6. The automorphism group is then the group $\mathrm{Aut}(D_4) \approx W(D_4) : S_3 \approx W(F_4)$. This is well known as triality of the D_4 symmetry. We also note that some of the orbits of $W(D_n)$ and the orbits of $W(B_n)$ have an identical number of vertices:

$$\left|(10\ldots0)_{D_n}\right| = \left|(10\ldots0)_{B_n}\right|, \left|(010\ldots0)_{D_n}\right| = \left|(010\ldots0)_{B_n}\right|,$$
$$,\ldots, \left|(00\ldots1000)_{D_n}\right| = \left|(00\ldots1000)_{B_n}\right|. \quad (8)$$

The orbit $(00\ldots01)_{B_n}$ is the union of two orbits of $W(D_n)$,

$$(00\ldots1)_{B_n} = (00\ldots10)_{D_n} \cup (00\ldots01)_{D_n}. \quad (9)$$

The group $W(B_n)$ admits the groups $W(B_{n-1})$ and $W(A_{n-1})$ as maximal subgroups. One of the interesting maximal subgroups of $W(B_n)$ is the dihedral group $W(I_2(h)) \approx D_h$ of order $2h$. Dihedral group D_h plays an important role in the study of two-dimensional quasicrystals with h-fold symmetry. The argument goes as follows. The simple roots of $W(B_n)$ decompose in such a way that the corresponding reflection generators $r_1, r_3, \ldots, r_{n-1}$ commute pairwise as well as the set $r_2, r_4, \ldots, r_n$ for even n. A similar decomposition can be done for odd n. Define now the generators R_1 and R_2 by $R_1 = r_1 r_3 \ldots r_{n-1}$ and $R_2 = r_2 r_4 \ldots r_n$ (Steinberg, 1951). It is easy to show that the generators R_1 and R_2 act as reflections on the simple roots of the Coxeter diagram $I_2(h)$ and the Coxeter element $R_1 R_2$ represents a rotation of order h in the plane spanned by the simple roots (Carter, 1972) of the Coxeter–Dynkin diagram $I_2(h)$. The Coxeter exponents $m_i \frac{\pi}{h} (i = 1, 2, \ldots, n)$ with $m_i = 1, 3, 5, \ldots, 2n - 1$ and the Coxeter number $h = 2n$ are useful in the determination of the Coxeter plane in which one can have a quasicrystallographic point symmetry of the projected set of points. The eigenvalues of the Cartan matrix in equation (2) can be written simply $\lambda_i = 2[1 - \cos(m_i \frac{\pi}{h})]$ (Coxeter, 1951). To study the quasicrystallography the plane determined by the eigenvectors corresponding to the pair of eigenvalues $2[1 - \cos(m_i \frac{\pi}{h})]$, $2[1 - \cos((h - m_i)\frac{\pi}{h})]$ is of interest. One can choose a convenient set of orthogonal unit vectors obtained from the eigenvectors of the Cartan matrix (Koca, Koca & Koc, 2014*a*; Koca, Koca & Al-Sawafi, 2014):

$$\hat{x}_i = \frac{1}{\sqrt{h\lambda_i}} \sum_j \frac{2\alpha_j}{(\alpha_j, \alpha_j)} X_{ji}, \quad (10)$$

where $\mathbf{X}_i$ is the eigenvector of the Cartan matrix given in equation (2) corresponding to the eigenvalue λ_i with a normalization where the last components of the eigenvectors all equal 1. The simple roots β_i, β_{n+1-i} of the dihedral group $I_2(\frac{h}{m_i})$,

$$\begin{aligned}\beta_i &= \sqrt{2}\Big[\sin\Big(\frac{m_i\pi}{2h}\Big)\hat{x}_i + \cos\Big(\frac{m_i\pi}{2h}\Big)\hat{x}_{n+1-i}\Big],\\ \beta_{n+1-i} &= \sqrt{2}\Big[\sin\Big(\frac{m_i\pi}{2h}\Big)\hat{x}_i - \cos\Big(\frac{m_i\pi}{2h}\Big)\hat{x}_{n+1-i}\Big],\\ &\Big(i = 1, 2, \ldots, \frac{n}{2}\Big),\end{aligned} \quad (11)$$

determine the principal planes in which the generators R_1 and R_2 act like reflection generators on the simple roots β_i, β_{n+1-i}. The Coxeter element R_1R_2 is represented by a block-diagonal matrix consisting of 2×2 matrices for even n in the space where the simple roots are represented by equation (11). This is equivalent to the statement that the lattice space is decomposed into two-dimensional spaces in which the dihedral group generated by R_1 and R_2 acts as a point group.

The root lattice of $W(B_n)$ is the simple cubic lattice which is invariant under the affine Coxeter group $W_a(B_n)$ that can be generated by adding a generator r_0 to the set of generators of the group $W(B_n)$. The generator r_0 represents a reflection with respect to the hyperplane bisecting the highest long root $\tilde{\alpha} = \omega_2 = l_1 + l_2$. Its action on an arbitrary vector is a translation.

A general vector of the root lattice then will be given by $p = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n, b_i \in \mathbb{Z}$ which can also be written in the weight space as $p = a_1\omega_1 + a_2\omega_2 + \ldots + 2a_n\omega_n, a_i \in \mathbb{Z}$. This indicates that a general vector of the lattice is the linear combinations of the unit vectors l_i with integer coefficients. This also implies that the root lattice of $W(B_n)$ is generated by its short roots. The primitive cell of the lattice can be chosen as the cube with the vertices

$$\begin{aligned}&0, l_i, l_i + l_j\,(i<j), l_i + l_j + l_k\,(i<j<k), \ldots,\\ &l_i + l_j + \ldots + l_k\,(i<j<\ldots<k).\end{aligned} \quad (12)$$

There are 2^n such cubes sharing the origin as a vertex. The Voronoi cells around the lattice points are congruent polytopes tiling the n-dimensional Euclidean space. The Voronoi cells of lattices are important in the theory of coding (Conway & Sloane, 1988). We denote the Voronoi cell around the origin by $V(0)$ and its structure is important for the canonical projection (strip projection, or cut-and-project technique) of the lattice points onto the Coxeter plane. Vertices of the Voronoi polytope $V(0)$ can be determined as the intersection of the hyperplanes surrounding the origin. They are the hyperplanes determined as the orbits of the fundamental weights

$$\begin{aligned}&\frac{\omega_1}{2} = \frac{l_1}{2}, \frac{\omega_2}{2} = \frac{l_1 + l_2}{2}, \ldots, \frac{\omega_{n-1}}{2} = \frac{l_1 + l_2 + \ldots l_{n-1}}{2},\\ &\omega_n = \frac{1}{2}(l_1 + l_2 + \ldots + l_n).\end{aligned} \quad (13)$$

The Voronoi polytope $V(0)$ is then a cube around the origin with the vertices (Conway & Sloane, 1982; Moody & Patera, 1992)

$$(00\ldots01)_{B_n} = \frac{1}{2}(\pm l_1 \pm l_2 \ldots \pm l_n). \quad (14)$$

Before we proceed further we emphasize here that the lattices generated by the affine Coxeter–Weyl group $W_a(C_n)$ are identical to the root and weight lattices of the Coxeter–Weyl group $W_a(D_n)$ (Conway & Sloane, 1988) since the short roots of $W(C_n)$ are identical to the root system of the group $W_a(D_n)$. The group $\mathrm{Aut}(D_n) \approx W(C_n)$ should be taken into account when the point symmetry of the two lattices is of concern. Therefore, in an n-dimensional Euclidean space with $n > 3$, one can construct five different lattices with affine Coxeter–Weyl groups, two for $W_a(A_n)$, two for $W_a(D_n)$, and the other is the simple cubic lattice described by $W_a(B_n)$. Of course if n coincides with the rank of the exceptional groups the number of lattices will be more than 5. We have to determine the components of a lattice vector in the principal planes defined by the pairs of unit vectors $(\hat{x}_1, \hat{x}_n), (\hat{x}_2, \hat{x}_{n-1}), \ldots$ where the plane determined by the vectors $(\hat{x}_1, \hat{x}_n)$ is known as the Coxeter plane. Note that for odd n one of the unit vectors is unpaired that represents the direction orthogonal to all Coxeter planes. The representation of the generators R_1 and R_2 of the dihedral subgroup can be put into block-diagonal matrices with 2×2 and/or 1×1 matrix entries. Some of the planes may display the crystallographic symmetries rather than the quasicrystallographic symmetries depending on the values of the Coxeter exponents m_i. The component of the simple cubic lattice vector in the basis of $\hat{x}_i$ is given by

$$p_i = \frac{1}{\sqrt{h\lambda_i}}\left(\sum_{j=1}^{n-1} a_j X_{ji} + 2a_n\right), a_i \in \mathbb{Z}. \quad (15)$$

This preliminary introduction to the hypercubic lattice and its symmetry group will be useful in the following sections where we study the eightfold, fivefold, tenfold and 12-fold symmetric quasicrystal structures induced by the projections of the lattices B_4, B_5 and B_6.

3. The Coxeter–Weyl groups $W(B_4)$, $W(B_5)$, $W(B_6)$ and projections of their polytopes into three-dimensional subspaces

The projected copies, in three-dimensional space, of the Voronoi cells of the root lattices of the Coxeter–Weyl groups $W(B_4)$, $W(B_5)$ and $W(B_6)$ are the rhombic dodecahedron, rhombic icosahedron and rhombic triacontahedron, respectively. This is quite well known in the literature but has never been presented in a systematic way. A similar work, not in the context of the affine Coxeter groups, has been reported by Kramer (1986).

3.1. The Coxeter–Weyl group $W(B_4)$ and projection of its fundamental polytopes into a three-dimensional subspace with octahedral symmetry

In the study of representations of the Lie groups the weight vectors $\omega_i (i = 1, 2, \ldots, n)$ play an important role and are called fundamental weights. Hereafter we will call those polytopes obtained as the orbits of the fundamental weights as the fundamental polytopes. The vertices of the fundamental polytopes of the group $W(B_4)$ are given by

$$\begin{aligned}(1000)_{B_4} &= \pm l_i,\\ (0100)_{B_4} &= \pm l_i \pm l_j \,(i \neq j),\\ (0010)_{B_4} &= \pm l_i \pm l_j \pm l_k \,(i \neq j \neq k),\\ (0001)_{B_4} &= \frac{1}{2}(\pm l_1 \pm l_2 \pm l_3 \pm l_4), i, j, k = 1, 2, 3, 4. \end{aligned} \tag{16}$$

These are well known four-dimensional polytopes with four-dimensional cubic symmetry. The first one represents the short roots of $W(B_4)$ which constitutes a four-dimensional octahedron with eight vertices whose facets (3-faces) are tetrahedra. The second polytope consists of 24 long roots as vertices and is known as the 24-cell with octahedral facets. Its full symmetry is the Coxeter–Weyl group $W(F_4)$ which embeds $W(B_4)$ as a subgroup with index 3. The third polytope with 32 vertices has two types of facets: tetrahedra and truncated octahedra. The last one is the four-dimensional cube with 16 vertices consisting of the cubic facets.

The subspace we are interested in here is a three-dimensional Euclidean space with the octahedral symmetry represented by the Coxeter–Weyl group $W(B_3)$ of order 48. Let us recall the isomorphism $W(B_3) \approx \mathrm{Aut}(D_3) \approx \mathrm{Aut}(A_3) = W(A_3) : \mathbb{Z}_2$. Here one can choose the tetrahedral symmetry as the Coxeter–Weyl group $W(A_3) = \langle r_1, r_2, r_3 \rangle$. The Dynkin diagram symmetry $\mathbb{Z}_2$ generated by γ which permutes the unit vectors as $l_1 \leftrightarrow l_3$ and $l_2 \leftrightarrow l_4$ extends the group to the octahedral symmetry. Using γ and equation (4) for the actions of the generators r_1, r_2 and r_3 it is clear that the vector $\frac{1}{2}(l_1 + l_2 + l_3 + l_4)$ is invariant under the octahedral group. To describe the three-dimensional Euclidean space where the octahedral group acts as the symmetry group, it is convenient to introduce a new set of orthonormal vectors defined by

$$\begin{aligned} t_0 &= \frac{1}{2}(l_1 + l_2 + l_3 + l_4), \quad t_1 = \frac{1}{2}(l_1 - l_2 + l_3 - l_4),\\ t_2 &= \frac{1}{2}(-l_1 + l_2 + l_3 - l_4), \quad t_3 = \frac{1}{2}(l_1 + l_2 - l_3 - l_4). \end{aligned} \tag{17}$$

When the vectors in equation (16) are expressed in terms of the vectors t_0, t_1, t_2 and t_3 it will be simpler to identify the three-dimensional vertices of the four-dimensional polytopes. The four-dimensional octahedron projects onto a cube in three dimensions with the vertices $\frac{1}{2}(\pm t_1 \pm t_2 \pm t_3)$. The 24-cell projected into the three-dimensional space represents two octahedra as well as one cuboctahedron. The polytope with 32 vertices is projected into one cube and two truncated tetrahedra. Under the octahedral symmetry the union of two truncated tetrahedra forms a non-regular polyhedron with 24 vertices. Its faces are made of equilateral triangles, squares and rectangles.

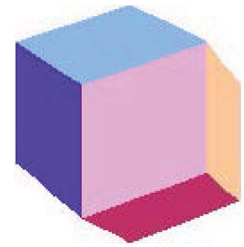


Figure 3
The rhombic dodecahedron.

The polytope $(0001)_{B_4}$ is more interesting since it constitutes the Voronoi cell of the four-dimensional cubic lattice. Two vectors $\pm\frac{1}{2}(l_1 + l_2 + l_3 + l_4)$ from the set are projected to the origin. When expressed in terms of the unit vectors t_0, t_1, t_2 and t_3 and the component of an arbitrary vector along t_0 is deleted they will decompose under the octahedral group as two orbits, one with the vertices $(\pm t_1, \pm t_2, \pm t_3)$ representing the vertices of an octahedron and the other with vertices $\frac{1}{2}(\pm t_1 \pm t_2 \pm t_3)$ representing a cube. Union of these two orbits represents a rhombic dodecahedron (Koca *et al.*, 2010) with 14 vertices as shown in Fig. 3.

3.2. The Coxeter-Weyl group $W(B_5)$ and projection of its fundamental polytopes into three-dimensional subspace with $W(H_2) \times C_2 \approx D_{5d}$ symmetry

One of the fundamental orbits of the group $W(B_5)$ is the polytope $(10000)_{B_5} = \pm l_i \,(i = 1, 2, \ldots, 5)$, which represents an octahedron in five-dimensional Euclidean space whose facets are 5-cells (4-simplexes). We will see that when it is projected into a three-dimensional space it represents a pentagonal antiprism where the subgroup $W(H_2) \times C_2 \approx D_{5d}$ acts as a point group. To understand this better we have to define a new set of orthonormal vectors $\hat{x}_i \,(i = 1, 2, \ldots, 5)$. The first four unit vectors $\hat{x}_i \,(i = 1, 2, \ldots, 4)$ are obtained by using the eigenvectors of the Cartan matrix of $W(A_4)$ and the fifth vector is chosen to be orthogonal to the rest (Koca, Koca & Koc, 2014*b*):

$$\begin{aligned} \hat{x}_1 &= \frac{1}{\sqrt{2(2+\sigma)}}(\alpha_1 + \tau\alpha_2 + \tau\alpha_3 + \alpha_4),\\ \hat{x}_2 &= \frac{1}{(2+\sigma)\sqrt{2}}(\alpha_1 - \sigma\alpha_2 + \sigma\alpha_3 - \alpha_4),\\ \hat{x}_3 &= \frac{1}{\sqrt{2(2+\tau)}}(\alpha_1 + \sigma\alpha_2 + \sigma\alpha_3 + \alpha_4),\\ \hat{x}_4 &= \frac{1}{(2+\tau)\sqrt{2}}(\alpha_1 - \tau\alpha_2 + \tau\alpha_3 - \alpha_4),\\ \hat{x}_5 &= \frac{1}{\sqrt{5}}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5). \end{aligned} \tag{18}$$

The unit vectors l_i can be expressed as a linear combination $l_i = \sum_{j=i}^{5} b_{ij}\hat{x}_j$ where the matrix B is given by

$$B=\frac{1}{\sqrt{10}}\begin{pmatrix} a & \tau & b & -\sigma & \sqrt{2}\\ b & \sigma & -a & -\tau & \sqrt{2}\\ 0 & -2 & 0 & 2 & \sqrt{2}\\ -b & \sigma & a & -\tau & \sqrt{2}\\ -a & \tau & -b & -\sigma & \sqrt{2}\end{pmatrix},$$

$$a=\sqrt{2+\tau},\ b=\sqrt{2+\sigma},\ \tau=\frac{1+\sqrt{5}}{2},\ \sigma=\frac{1-\sqrt{5}}{2}. \qquad (19)$$

For their relevance to the projection technique of the five-dimensional lattice, we now discuss the projections of only two polytopes $(10000)_{B_5}=\pm l_i\,(i=1,2,\ldots,5)$ and $(00001)_{B_5}=\frac{1}{2}(\pm l_1\pm l_2\pm l_3\pm l_4\pm l_5)$ into the three-dimensional space described by the symmetry group $W(H_2)\times C_2\approx D_{5d}$. The other fundamental polytopes lead to some quasiregular polyhedra in three-dimensional space and are not of particular interest here. The group D_{5d} is generated by the group elements $R_1=r_1r_3$, $R_2=r_2r_4$ and $R_3=(r_1r_2r_3r_4r_5)^5$. Here R_1 and R_2 generate the dihedral group $W(H_2)\approx D_5$ of order 10 and $R_3=-I$ where I is the 5×5 unit matrix in the l_i basis. The centre of the group $W(B_5)$ is represented by the elements $C_2=\{I,-I\}$ and therefore it commutes with all the elements of the group. To choose a three-dimensional subspace there are two options: either the space spanned by $(\hat{x}_1,\hat{x}_4,\hat{x}_5)$ or $(\hat{x}_2,\hat{x}_3,\hat{x}_5)$. Let us choose the three-dimensional space defined by the first set of unit vectors. The set of vectors $\pm l_i$ forms a single orbit under the group D_{5d}. The polytope $(10000)_{B_5}=\pm l_i$ projected into three-dimensional space represents a pentagonal antiprism as shown in Fig. 4.

The polytope $(00001)_{B_5}=\frac{1}{2}(\pm l_1\pm l_2\pm l_3\pm l_4\pm l_5)$ represents the Voronoi cell of the five-dimensional cubic lattice. The 32 vertices decompose under the group D_{5d} as 32 = 2 + 10 + 20. The first orbit of size 2 represents the vectors $\pm\frac{1}{2}(l_1+l_2+l_3+l_4+l_5)$. Each vector is invariant under the dihedral group D_5 but changed to each other under the elements of the centre C_2. The next orbit of size 10 consists of the vectors like $\pm\frac{1}{2}(-l_1+l_2+l_3+l_4+l_5)$ with one or four negative signs. They also constitute a pentagonal antiprism like the one in Fig. 4. The orbit of size 20 consists of the vectors with two negative and/or three negative signs. The union of two orbits 2 + 20 constitutes a rhombic icosahedron as shown in Fig. 5.

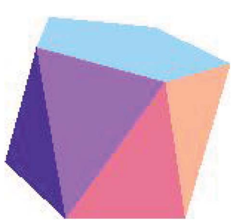


Figure 4
The pentagonal antiprism.

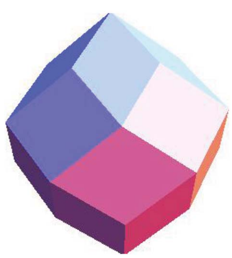


Figure 5
The rhombic icosahedron with symmetry D_{5d}.

3.3. The Coxeter–Weyl group $W(B_6)$ and projections of some of its fundamental polytopes into a three-dimensional subspace with $W(H_3)\approx I_h$ symmetry

The icosahedral symmetry is the one that describes some quasicrystal structures in three dimensions. Here we first discuss how the icosahedral symmetry, the Coxeter group $W(H_3)\approx I_h$, can be obtained as a subgroup of the Coxeter–Weyl group $W(B_6)$. We have already discussed in §2 that one of the maximal subgroups of the group $W(B_n)$ is the group $W(D_n)$. Our interest is then the Coxeter–Weyl group $W(D_6)$. We introduce the generators $R_1=r_1'r_5'$, $R_2=r_2'r_4'$ and $R_3=r_3'r_6'$. They generate the Coxeter group $W(H_3)$ (Shcherbak, 1988; Koca *et al.*, 1998, 2001). Note that in terms of the generators of the group $W(B_6)$ they can be written as $R_1=r_1r_5$, $R_2=r_2r_4$ and $R_3=r_3r_6r_5r_6$. They satisfy the relations

$$R_1^{\,2}=R_2^{\,2}=R_3^{\,2}=(R_1R_3)^2=(R_1R_2)^3=(R_2R_3)^5=1, \qquad (20)$$

leading to the usual generation relations of the icosahedral group $W(H_3)\approx A_5\times C_2=I_h$. The generators of the icosahedral group are 6×6 matrices in the space of orthonormal vectors l_i and constitute a reducible representation of the icosahedral group. They can be transformed into block-diagonal forms of 3×3 matrices which act on two sets of orthonormal vectors $(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ and $(\hat{x}_1{}',\hat{x}_2{}',\hat{x}_3{}')$. Each block represents a different 3×3 irreducible matrix representation.

The block-diagonal form of the generators of the Coxeter group $W(H_3)$ induces the relation

$$\begin{pmatrix} l_1\\ l_2\\ l_3\\ l_4\\ l_5\\ l_6\end{pmatrix}=\frac{1}{\sqrt{2(2+\tau)}}\begin{pmatrix} -1 & -\tau & 0 & -\tau & 1 & 0\\ 1 & -\tau & 0 & \tau & 1 & 0\\ 0 & -1 & -\tau & 0 & -\tau & 1\\ 0 & -1 & \tau & 0 & -\tau & -1\\ -\tau & 0 & -1 & 1 & 0 & -\tau\\ \tau & 0 & -1 & -1 & 0 & -\tau\end{pmatrix}\begin{pmatrix} \hat{x}_1\\ \hat{x}_2\\ \hat{x}_3\\ \hat{x}_1{}'\\ \hat{x}_2{}'\\ \hat{x}_3{}'\end{pmatrix}. \qquad (21)$$

Before proceeding further we note that one can also construct the generators of the Coxeter group $W(H_3)$ as $R_1=r_1'r_6'$, $R_2=r_2'r_4'$ and $R_3=r_3'r_5'$ from Fig. 2 for $n=6$. No elements of the group $W(D_6)$ transform this icosahedral group to the one given in equation (20). However, these two icosahedral groups can be transformed to each other by the Dynkin diagram symmetry generator of the Coxeter–Dynkin diagram of D_6. The Dynkin diagram symmetry generator $\gamma: r_5'\leftrightarrow r_6'$ interchanges the last two generators of the diagram D_6 and can be associated with the $r_6=\gamma$ generator of the group $W(B_6)$. Since $r_5'=r_5$ and $r_6'=r_6r_5r_6$ then the icosahedral group defined in equation (20) is conjugate to the one defined above in the group $\mathrm{Aut}(D_6)\approx W(B_6)$.

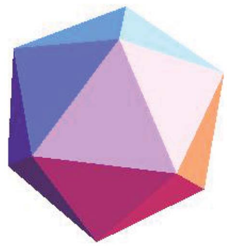


Figure 6
An icosahedron projected from the six-dimensional octahedron $(100000)_{B_6}$.

For a projection into three-dimensional space one can use either the first three components or the last three components of the unit vectors l_i. Now we discuss the projections of certain $W(B_6)$ polytopes into three-dimensional space with a residual icosahedral symmetry. The 12 vertices of the polytope $(100000)_{B_6} = \pm l_i$ are the short roots of the group $W(B_6)$ representing the vertices of a six-dimensional octahedron. The projected copy in three-dimensional space turns out to be an icosahedron represented by the vectors of norm $\frac{1}{\sqrt{2}} \approx 0.707$ as shown in Fig. 6.

The long roots of the group $W(B_6)$ are the vertices of the polytope $(010000)_{B_6}$ which are given by the 60 vectors $(010000)_{B_6} = \pm l_i \pm l_j \, (i \neq j)$. When they are projected into three-dimensional space they represent two copies of icosidodecahedra with 30 vertices each, one copy is expanded with respect to the other by a factor of τ. Their actual norms in three dimensions are

$$\left(\frac{2}{2+\tau}\right)^{1/2} \approx 0.743 \text{ and } \left(\frac{2}{2+\sigma}\right)^{1/2} \approx 1.203.$$

The icosidodecahedron is a polyhedron with 30 vertices, 32 faces (20 triangles + 12 pentagons) and 60 edges. One of the icosidodecahedra is depicted in Fig. 7.

The polytope $(000001)_{B_6} = \frac{1}{2}(\pm l_1 \pm l_2 \pm l_3 \pm l_4 \pm l_5 \pm l_6)$ represents the Voronoi cell of the six-dimensional cubic lattice with 64 vertices. They decompose into sets with even $(-)$ sign and odd $(-)$ sign representing two orbits of $W(D_6)$ as mentioned in §2. We will explicitly demonstrate that each orbit of $W(D_6)$ with 32 vertices decomposes as $32 = 20 + 12$. Normally, the orbits of size 20 and 12 represent a dodecahedron and an icosahedron, respectively. However, as we will discuss below, the situation here is such that the dodecahedron (orbit I) and icosahedron (orbit III) form a rhombic triacontahedron, a Catalan solid dual to the icosidodecahedron (Koca *et al.*, 2010). The union of the icosahedron (orbit II) and dodecahedron (orbit IV) represents a star dodecahedron. Below we give explicit decomposition of the vertices of the six-dimensional cube under the Coxeter group $W(H_3)$. The first two orbits are the decomposition of the vertices with even $(-)$ sign of vectors and the next two are the vertices with odd $(-)$ sign.

Orbit I: dodecahedron (I)

$$\begin{aligned}
&\pm\tfrac{1}{2}(l_1+l_2+l_3+l_4+l_5+l_6),\ \pm\tfrac{1}{2}(l_1+l_2+l_3-l_4-l_5+l_6),\\
&\pm\tfrac{1}{2}(-l_1+l_2+l_3-l_4+l_5+l_6),\ \pm\tfrac{1}{2}(-l_1-l_2+l_3-l_4-l_5+l_6),\\
&\pm\tfrac{1}{2}(-l_1-l_2-l_3-l_4+l_5+l_6),\ \pm\tfrac{1}{2}(l_1+l_2+l_3-l_4+l_5-l_6),\\
&\pm\tfrac{1}{2}(-l_1+l_2+l_3+l_4-l_5+l_6),\ \pm\tfrac{1}{2}(l_1-l_2+l_3-l_4+l_5+l_6),\\
&\pm\tfrac{1}{2}(-l_1+l_2-l_3-l_4-l_5+l_6),\ \pm\tfrac{1}{2}(-l_1-l_2+l_3-l_4+l_5-l_6).
\end{aligned} \tag{22a}$$

Orbit II: icosahedron (II)

$$\begin{aligned}
&\pm\tfrac{1}{2}(-l_1-l_2+l_3+l_4+l_5+l_6),\ \pm\tfrac{1}{2}(l_1-l_2-l_3-l_4-l_5+l_6),\\
&\pm\tfrac{1}{2}(-l_1+l_2-l_3-l_4+l_5-l_6),\ \pm\tfrac{1}{2}(-l_1-l_2+l_3+l_4-l_5-l_6),\\
&\pm\tfrac{1}{2}(l_1-l_2-l_3+l_4+l_5+l_6),\ \pm\tfrac{1}{2}(-l_1+l_2-l_3+l_4+l_5+l_6).
\end{aligned} \tag{22b}$$

Orbit III: icosahedron (III)

$$\begin{aligned}
&\pm\tfrac{1}{2}(l_1+l_2+l_3+l_4+l_5-l_6),\ \pm\tfrac{1}{2}(l_1+l_2+l_3+l_4-l_5+l_6),\\
&\pm\tfrac{1}{2}(l_1+l_2+l_3-l_4+l_5+l_6),\ \pm\tfrac{1}{2}(-l_1+l_2+l_3-l_4-l_5+l_6),\\
&\pm\tfrac{1}{2}(-l_1-l_2+l_3-l_4+l_5+l_6),\ \pm\tfrac{1}{2}(l_1-l_2+l_3-l_4+l_5-l_6).
\end{aligned} \tag{22c}$$

Orbit IV: dodecahedron (IV)

$$\begin{aligned}
&\pm\tfrac{1}{2}(l_1-l_2+l_3-l_4-l_5+l_6),\ \pm\tfrac{1}{2}(-l_1+l_2-l_3-l_4+l_5+l_6),\\
&\pm\tfrac{1}{2}(-l_1-l_2+l_3-l_4-l_5-l_6),\ \pm\tfrac{1}{2}(-l_1-l_2-l_3+l_4+l_5+l_6),\\
&\pm\tfrac{1}{2}(l_1-l_2-l_3-l_4-l_5-l_6),\ \pm\tfrac{1}{2}(-l_1+l_2+l_3-l_4+l_5-l_6),\\
&\pm\tfrac{1}{2}(-l_1-l_2+l_3+l_4-l_5+l_6),\ \pm\tfrac{1}{2}(l_1-l_2-l_3-l_4+l_5+l_6),\\
&\pm\tfrac{1}{2}(-l_1+l_2-l_3-l_4-l_5-l_6),\ \pm\tfrac{1}{2}(-l_1-l_2+l_3+l_4+l_5-l_6).
\end{aligned} \tag{22d}$$

To see why they represent dodecahedra and icosahedra we just replace the vectors l_i by their first three components in equation (21). However, when we take the union of orbit I and orbit III we obtain the rhombic triacontahedron as shown in Fig. 8. The orbit II and the orbit IV form a dodecahedral star which is depicted in Fig. 9.

Figure 7
Icosidodecahedron projected from the six-dimensional polytope $(010000)_{B_6}$.

The vectors representing the vertices of the icosahedra and dodecahedra have the following norms in decreasing order:

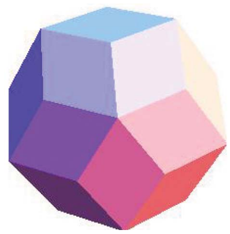


Figure 8
The rhombic triacontahedron obtained as part of the projection of the six-dimensional cube.

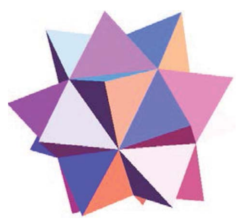


Figure 9
The dodecahedral star projected from the six-dimensional cube.

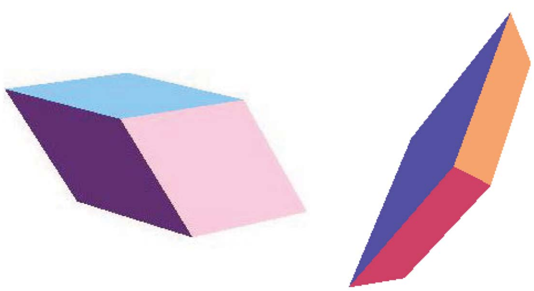


Figure 10
The acute and obtuse rhombohedra generated by the vectors l_i.

$$\begin{aligned} N(\text{icosahedron III}) &= \frac{\tau}{\sqrt{2}} \approx 1.144, \\ N(\text{dodecahedron I}) &= \sqrt{0.3(2+\tau)} \approx 1.042, \\ N(\text{dodecahedron IV}) &= \sqrt{0.3(2+\sigma)} \approx 0.644, \\ N(\text{icosahedron II}) &= \frac{-\sigma}{\sqrt{2}} \approx 0.438. \end{aligned} \tag{23}$$

These are exactly the same results obtained earlier by Conway & Knowles (1986) without referring to the overall group structure of the B_6 lattice. Note also that

$$\begin{aligned} N(\text{icosahedron III})/N(\text{icosahedron II}) &= \tau^2, \\ N(\text{dodecahedron I})/N(\text{dodecahedron IV}) &= \tau. \end{aligned} \tag{24}$$

Since the icosahedron obtained from the short roots has a norm 0.707, these three icosahedra follow the ratio $(1 : \tau : \tau^2)$ up to an overall scale factor. We also note in passing that the vectors $\pm l_i$, when projected into three-dimensional subspace, form rhombohedra when taken in groups of three. For example the sets of vectors (l_1, l_2, l_3) and (l_4, l_5, l_6) form an acute and an obtuse rhombohedra, respectively, as shown in Fig. 10.

4. Projections of the lattices of the Coxeter–Weyl groups $W(B_4)$, $W(B_5)$ and $W(B_6)$ into subspaces

The technique which we developed for the projection of the polytopes can now be applied to the lattice points. We will study each case separately. We decompose the nD space into two subspaces $E_\parallel$ and $E_\perp$, where $E_\parallel$ represents the subspace into which the lattice points are to be projected and the subspace $E_\perp$ is the complementary orthogonal subspace. The shift of the Voronoi cell $V(0)$ along the space $E_\parallel$ creates an open strip and the projection of the Voronoi polytope into the subspace $E_\perp$ determines a region $\mathbf{K}$. It has been shown with a number of examples that the set of lattice points projected from the open strip onto the subspace $E_\parallel$ determines a quasicrystallographic structure (Kalugin *et al.*, 1985; Elser, 1985; Elser & Henley, 1985). We will discuss the applications of this procedure in the following subsections.

4.1. Projection of the lattice points of B_4 into a two-dimensional space

We will show here that the projected set of lattice points displays a quasicrystal with eightfold symmetry. The components of a lattice vector $p = a_1\omega_1 + a_2\omega_2 + a_3\omega_3 + 2a_4\omega_4$, $a_i \in \mathbb{Z}$ along the unit vectors $\hat{x}_i$ are given as follows:

$$\begin{aligned} p_1 &= \frac{1}{2\sqrt{2(2-\sqrt{2+\sqrt{2}})}}\left(\sqrt{2-\sqrt{2}}a_1 + \sqrt{2}a_2 + \sqrt{2+\sqrt{2}}a_3 + 2a_4\right) \\ p_4 &= \frac{1}{2\sqrt{2(2+\sqrt{2+\sqrt{2}})}}\left(-\sqrt{2-\sqrt{2}}a_1 + \sqrt{2}a_2 - \sqrt{2+\sqrt{2}}a_3 + 2a_4\right) \\ p_2 &= \frac{1}{2\sqrt{2(2-\sqrt{2-\sqrt{2}})}}\left(-\sqrt{2+\sqrt{2}}a_1 - \sqrt{2}a_2 + \sqrt{2-\sqrt{2}}a_3 + 2a_4\right) \\ p_3 &= \frac{1}{2\sqrt{2(2+\sqrt{2-\sqrt{2}})}}\left(\sqrt{2+\sqrt{2}}a_1 - \sqrt{2}a_2 - \sqrt{2-\sqrt{2}}a_3 + 2a_4\right). \end{aligned} \tag{25}$$

Let us assume that $E_\parallel = (\hat{x}_1, \hat{x}_4)$ and $E_\perp = (\hat{x}_2, \hat{x}_3)$. When the circumsphere of the Voronoi cell $V(0) = (0001)_{B_4}$ is projected onto the plane $E_\perp = (\hat{x}_2, \hat{x}_3)$, it determines a disc of radius $R_0 = \max({p_2}^2 + {p_3}^2)$. This defines a cylinder in the lattice constraining the integers a_i. When the shifted Voronoi cell $V(\omega_4) = (0001)_{B_4} + \omega_4$ is projected into the subspace $E_\perp = (\hat{x}_2, \hat{x}_3)$ the components (p_1, p_4) now determine the quasicrystal structure with eightfold symmetry as shown in Fig. 11. It consists of squares and rhombi with angles of 45°. Similar structures were obtained in a recent paper (Jagannathan & Duneau, 2014). A quasicrystal structure with eightfold symmetry has been observed in rapidly solidified $Cr_5Ni_3Si_2$ and $V_{15}Ni_{10}Si$ alloys (Wang *et al.*, 1987), and can be represented by quasicrystal structures like that in Fig. 11.

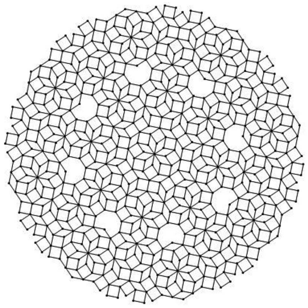


Figure 11
The quasicrystal structure obtained from the four-dimensional cubic lattice.

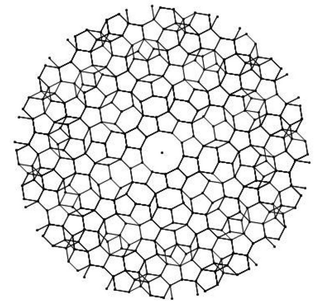


Figure 13
Tenfold symmetric quasicrystal structure from the five-dimensional cubic lattice.

4.2. Projection of the lattice B_5 into a two-dimensional subspace

Projection of the five-dimensional cubic lattice onto a two-dimensional plane with fivefold symmetry has been discussed previously (de Bruijn, 1981) with no reference to the Coxeter group $W(B_5)$. As we have pointed out in §2, the Coxeter number of the group $W(B_5)$ is 10 and has a dihedral subgroup D_{10} of order 20 with a cyclic subgroup of order 10 leading to a tenfold symmetry. In this section we will study the projections with both symmetries. The fivefold symmetric projection of the lattice can be obtained by taking the set of orthogonal vectors given by equation (18). Since we project the lattice on a plane where the dihedral subgroup is taken from the subgroup $W(A_4) \subset W(B_5)$ whose Coxeter number is 5, this leads to a fivefold symmetric structure. The components of a general root lattice vector $p = (a_1\omega_1 + a_2\omega_2 + a_3\omega_3 + a_4\omega_4 + 2a_5\omega_5)$, $a_i \in \mathbb{Z}$ can be determined by taking the scalar product of the lattice vector with the unit vectors in (18). Projection of the Voronoi cell $V(0)$ into the space defined by the unit vectors $(\hat{x}_2, \hat{x}_3, \hat{x}_5)$ determines the window $\boldsymbol{K}$ which constrains the lattice vectors to be projected onto the plane $(\hat{x}_1, \hat{x}_4)$. When the shifted vector $V(\omega_5) = (00001)_{B_5} + \omega_5$ is projected into the circumsphere of the rhombic icosahedron in the subspace $(\hat{x}_2, \hat{x}_3, \hat{x}_5)$, the distribution of the aperiodic lattice structure in the Coxeter plane $(\hat{x}_1, \hat{x}_4)$ is depicted in Fig. 12. The $\hat{x}_5$ direction in this case preserves the translational invariance.

The tenfold symmetric quasicrystal structure by projection of the root lattice of the group $W(B_5)$ can be obtained by using the orthonormal vectors in equation (10). The unit vectors in equation (10) follow the sequence of the Coxeter exponents $m_1 = 1, m_2 = 3, m_3 = 5, m_4 = 7, m_5 = 9$. To distinguish this set of vectors from those in equation (18) we will label them as $x_i{}' \,(i = 1, 2, 3, 4, 5)$. This sequence implies that the pairs $(\hat{x}_1{}', \hat{x}_5{}')$ and $(\hat{x}_2{}', \hat{x}_4{}')$ determine the principal planes with the first pair being the Coxeter plane; the first one can be taken as $E_\parallel$ and the second pair together with the unit vector $\hat{x}_3{}'$ defines the subspace $E_\perp$. The components of the lattice vectors in the planes and the orthogonal direction are given by

$$E_\parallel : \left\{ \begin{aligned} p_1 &= \frac{1}{\sqrt{10}\sqrt{2-\sqrt{2+\tau}}}(-\sigma a_1 + \sqrt{2+\sigma} a_2 \\ &\quad + \tau a_3 + \sqrt{2+\tau} a_4 + 2a_5), \\ p_5 &= \frac{1}{\sqrt{10}\sqrt{2+\sqrt{2+\tau}}}(-\sigma a_1 - \sqrt{2+\sigma} a_2 \\ &\quad + \tau a_3 - \sqrt{2+\tau} a_4 + 2a_5) \end{aligned} \right\},$$

$$E_\perp : \left\{ \begin{aligned} p_2 &= \frac{1}{\sqrt{10}\sqrt{2-\sqrt{2+\sigma}}}(-\tau a_1 - \sqrt{2+\tau} a_2 \\ &\quad + \sigma a_3 + \sqrt{2+\sigma} a_4 + 2a_5), \\ p_4 &= \frac{1}{\sqrt{10}\sqrt{2+\sqrt{2+\sigma}}}(-\tau a_1 + \sqrt{2+\tau} a_2 \\ &\quad + \sigma a_3 - \sqrt{2+\sigma} a_4 + 2a_5), \\ p_3 &= \frac{1}{\sqrt{5}}(a_1 - a_3 + a_5) \end{aligned} \right\}. \quad (26)$$

The quasicrystal structure with tenfold symmetry from the projection of the root lattice of the Coxeter group $W(B_5)$ is shown in Fig. 13.

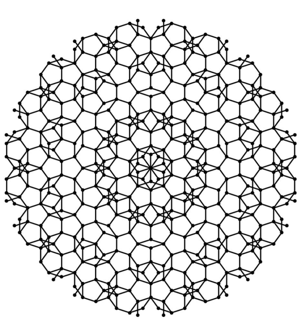


Figure 12
Fivefold symmetric structure from projection of the $W(B_5)$ root lattice.

4.3. Projection of the root lattice of B_6 into two-dimensional and three-dimensional subspaces

The Coxeter number of the group $W(B_6)$ is 12. Therefore it is quite natural to expect a 12-fold symmetric quasicrystal structure in a plane from the projection of the root lattice of the six-dimensional cubic lattice. We have studied projection of the six-dimensional cubic lattice in another publication (Koca, Koca & Koc, 2014*a*; Koca, Koca & Al-Sawafi, 2014) using the formula (10) for the group $W(B_6)$ and obtained the quasicrystal structure in Fig. 14.

Fig. 14 shows that the dodecagonal tiling displayed here is different to the usual tiling which is based only on the triangle

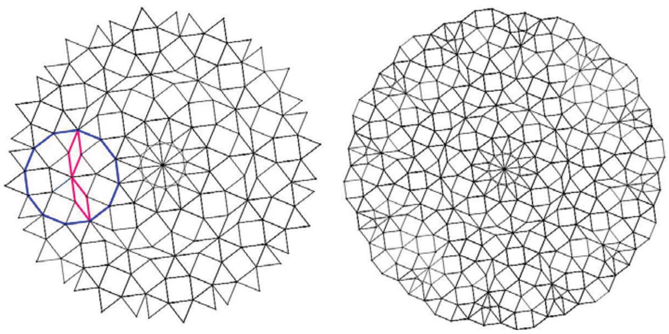


Figure 14
Twelvefold symmetric quasicrystal structure from the six-dimensional cubic lattice.

and square tiles. It also involves rhombi in addition to square and triangle tiles. Such a structure has been recently observed in the dodecagonal quasicrystal formation of $BaTiO_3$ (Förster *et al.*, 2013).

The projection of the six-dimensional cubic lattice into three-dimensional subspace has been discussed previously in various papers without taking into account the detailed structure of the Coxeter group $W(B_6)$ (Duneau & Katz, 1985; Elser, 1985; Conway & Knowles, 1986). In §3.3 we studied explicitly the icosahedral subgroup $W(H_3)$ of the Coxeter group $W(B_6)$ and showed that the space can be decomposed as $E_\parallel$ and $E_\perp$ defined by the unit vectors $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and $(\hat{x}'_1, \hat{x}'_2, \hat{x}'_3)$, respectively. A general root lattice vector can be written as

$$p = \frac{1}{\sqrt{2(2+\tau)}}[(n_1+n_2\tau)\hat{x}_1 + (n_3+n_4\tau)\hat{x}_2 + (n_5+n_6\tau)\hat{x}_3 + \tau(n_1+n_2\sigma)\hat{x}'_1 + \tau(n_3+n_4\sigma)\hat{x}'_2 + \tau(n_5+n_6\sigma)\hat{x}'_3], \quad (27)$$

where the integers n_i are given by

$$\begin{aligned} &n_1 = -a_1, n_2 = -a_5, n_3 = -(a_3+2a_4+2a_5+2a_6), \\ &n_4 = -(a_1+2a_2+2a_3+2a_4+2a_5+2a_6), \\ &n_5 = -(a_5+2a_6), n_6 = -a_3. \end{aligned} \quad (28)$$

A similar formula to equation (27) involving just the coefficients of the unit vectors $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$ was obtained previously from another consideration (Rokhsar *et al.*, 1987).

The root vectors $\beta_i\,(i = 1, 2, 3)$ of the Coxeter diagram of $W(H_3)$ can be determined as $\beta_1 = -\sqrt{2}\hat{x}_1$, $\beta_2 = \frac{1}{\sqrt{2}}(\hat{x}_1 + \sigma\hat{x}_2 + \tau\hat{x}_3)$, $\beta_3 = -\sqrt{2}\hat{x}_3$. Note that the pairs of vectors (β_1, β_3), (β_1, β_2) and (β_2, β_3) determine the twofold, threefold and fivefold symmetry planes, respectively. By choosing suitable orthogonal vectors in these planes such as $(\hat{x}_1, \hat{x}_3)$, $(\hat{y}_1, \hat{y}_2)$ and $(\hat{z}_1, \hat{z}_2)$ where

$$\begin{aligned} &\hat{y}_1 = \frac{1}{2}(-\hat{x}_1 + \sigma\hat{x}_2 + \tau\hat{x}_3),\ \hat{y}_2 = \frac{-1}{2\sqrt{3}}(3\hat{x}_1 + \sigma\hat{x}_2 + \tau\hat{x}_3), \\ &\hat{z}_1 = \frac{1}{2}(\tau\hat{x}_1 - \hat{x}_2 + \sigma\hat{x}_3),\ \hat{z}_2 = \frac{1}{2\sqrt{2+\tau}}(\hat{x}_1 + \sigma\hat{x}_2 + (2+\tau)\hat{x}_3), \end{aligned} \quad (29)$$

one can project the lattice vectors in equation (27) onto these planes provided the $E_\perp$ components of the vectors remain in the window $\boldsymbol{K}$ determined by the projection of the circumsphere of the Voronoi cell of the six-dimensional cube.

The distributions of the lattice points are displayed in Figs. 15, 16 and 17.

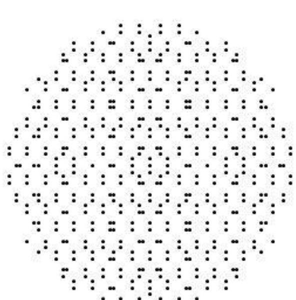


Figure 15
Twofold symmetry from projection of the B_6 lattice.

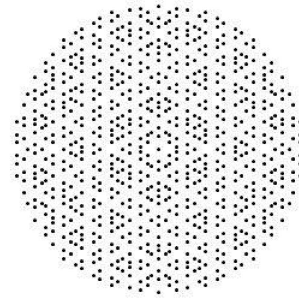


Figure 16
Threefold symmetry from projection of the B_6 lattice.

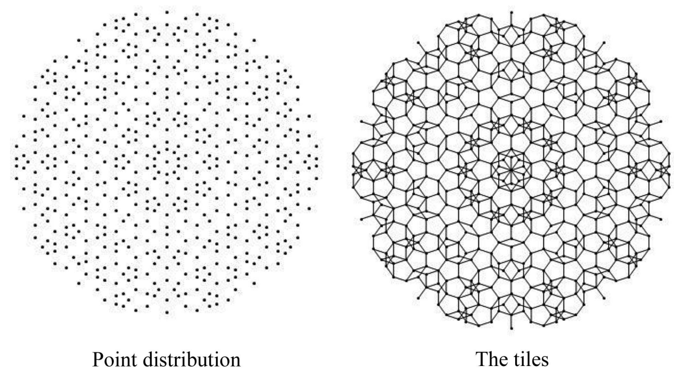


Figure 17
Fivefold symmetry from projection of the B_6 lattice.

5. Conclusions

We presented a systematic analysis of the higher-dimensional lattice projection technique with an emphasis on the group-theoretical structure of the *nD* cubic lattices. It is proposed that the Coxeter number h of the Coxeter–Weyl group $W(B_n)$ plays a crucial role in the determination of the dihedral subgroup D_h of the group $W(B_n)$, that in turn determines the symmetry of the quasicrystal structure in the Coxeter plane. The eigenvalues and eigenvectors of the Cartan matrix (Gram matrix) lead to the correct choice of the Coxeter plane onto which the lattice is projected. We reproduce the earlier results obtained from the lattice projections. In addition, in two cases we obtained new results: the tenfold symmetric quasicrystallography from $W(B_5)$ and the 12-fold symmetric quasicrystal structure from $W(B_6)$ lead to some novel structures. In particular the tiling displayed in Fig. 14 is compatible with a recent experiment with 12-fold symmetry (Förster *et al.*, 2013). The technique developed here can be extended to any higher-dimensional lattice described by the whole series of affine Coxeter groups. Projections of these lattices by choosing the $E_\perp$ space as the projected sets of the Voronoi cells have been studied in the literature previously. Here we presented an alternative approach by projecting the circumsphere of the Voronoi cell into the $E_\perp$ space to confront two different techniques.

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